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Phase transitions induced by thermal fluctuations

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Abstract. The present study generalizes the model of extended stochastic systems with a field-dependent kinetic coefficient [M. Ibanes, J. Garcia-Ojalvo, R. Toral, J.M. Sancho, Phys. Rev. Lett. **87**, 020601 (2001)] to systems with symmetric and asymmetric bistable potentials. It is found that in systems with a relaxational flow and a symmetric local potential, reentrant phase transitions can be observed. In the case of an asymmetric local potential, a hysteresis-like behaviour in the order parameter appears. It is shown that such phase transitions can be controlled by the constant that governs relaxation flow, noise intensity and spatial coupling intensity.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 05.10.Gg Stochastic analysis methods (Fokker-Planck, Langevin, etc.) – 64.60.-i General studies of phase transitions

1 Introduction

It is well known that nonlinear systems that exhibit disordered behaviour in the absence of fluctuations can be organized to sustain ordered states when an additional amount of noise is added [1,5]. In recent decades, many studies have focused on investigations of noise-induced phenomena which demonstrate a counterintuitive role for fluctuations, leading to self-organization effects, such as noise-induced transitions in zero-dimensional systems (see Ref. [1] and citations therein), stochastic resonance [2], noise-induced spatial patterns and phase transitions [3–5], and phase transitions induced by cross-correlations of noises [6,7]. As will be discussed in this paper, one of the most interesting effects is an ordering phase transition in extended systems, wherein the ordered phase (in a thermodynamic sense) only results if a randomly fluctuating source is introduced into the dynamical system, which must possess spatial degrees of freedom.

Most of the works concerning the above phenomena have focused on the problems concerning the influence of external noise. Analytically, numerically, and experimentally it was found that an external noise source only plays an organizing role if its amplitude depends on the field variable (see Refs. [1,5,8,9]). This result was explained as follows: in systems with fluctuations having a bounded frequency spectrum, the ordered phase exists for a particular range of the system parameters such as the control parameter, the noise intensity, and the intensity of spatial coupling (see Refs. [5,7,8]). Such reentrant phase transitions correspond to cases wherein an increase in one of the above parameters leads to an ordering dynamics once a first critical threshold is crossed, but after a second threshold is passed, the system becomes disordered. The above reentrance appears as a result of the combined effect of the nonlinearity of the system, the spectrally variant nature of the noise, and the spatial coupling. If the properties of the external fluctuations can be controlled in experiments, then one can govern the system dynamics by modifying the noise intensity, its spectral bandwidth, or the correlation properties of the fluctuations. From a fundamental point of view, such effects have a dynamic origin: in the short-time limit, external fluctuations destabilize the disordered homogeneous state. An analytic description of extended systems subject to external fluctuations is provided with an approximate, known stationary distribution function.

Recently, a new class of phase transitions was found [10] in which fluctuations do not lead to instability in the disordered phase (homogeneous mixture). Here, the ordered (separated) phase appears due to the balance between the relaxing forces moving the system to the homogeneous state, and field-variable dependent fluctuations pulling the system away from the disordered state. This mechanism belongs to a set of entropy driven phase transitions, which are the extension of noise-induced unimodal-bimodal transitions in zero-dimensional systems [1]. The origin of such phase transitions is due to a change in the form of the nonequilibrium potential [10–12]. The novelty of this phase transition lies in the fact that it arises

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entirely from an energy functional-like relaxation dynamics. Its occurrence indicates the presence of two elements in the stochastic dynamics: a field-dependent kinetic coefficient and the existence of a fluctuation dissipation relation. It allows the interpretation of the corresponding fluctuations as internal noise with intensity proportional to the bath temperature. For such a class of stochastic systems, the corresponding distribution function, free energy and an associated effective potential are known exactly. Therefore, noise-induced phase transitions can be analyzed without any dynamic reference.

Analytic work on these models has been fairly limited. Phase transitions of this kind were studied under special conditions with relaxation flow and a quadratic form of the local potential. It was found that an increase in the noise intensity, which plays the role of the bath temperature, leads to the formation of an ordered phase [10,11]. Our aims in this paper are therefore the following: (1) to generalize the above picture of entropy-driven phase transitions by considering systems with field-dependent relaxational dynamics and a local potential of a more complicated form, and (2) to investigate the possible reconstruction of the system states by controlling the properties of the relaxation flow and the noise intensity. Using mean field theory, we will describe the mechanisms of reentrant phase transitions in stochastic systems for which the local potential has a symmetrical form. For systems with an asymmetric local potential, we explore phase transitions which display hysteresis.

The paper is organized as follows. In Section 2, we review the relaxational model and the theoretical approach which are to be used in our considerations. Section 3 is devoted to the analysis of phase transitions induced by multiplicative noise in systems with both symmetrical and nonsymmetrical forms of the local potential. We conclude with a short summary in Section 4.

2 Theoretical tools

Let us consider the system described by the real scalar field $x = x(\mathbf{r}, t)$ which obeys the Langevin equation

$$\frac{\partial x}{\partial t} = -M(x)\frac{\delta}{\delta x}\mathcal{F}[x] + \zeta(\mathbf{r}, t). \tag{1}$$

In the deterministic case, equation (1) corresponds to the relaxation flow in the potential $\mathcal{F}[x]$ and the field-dependent kinetic coefficient $M(x)$ is interpreted as the x-dependent mobility. The free energy functional $\mathcal{F}$ is assumed to be of the form

$$\mathcal{F}[x] = \int d\mathbf{r}\left(V(x) + \frac{D}{4d}(\nabla x)^2\right), \quad \nabla \equiv \frac{\partial}{\partial \mathbf{r}}, \tag{2}$$

where $V(x)$ is the local potential, d is the spatial dimension, and D is a multiplicative constant. Regardless of the form of the local potential $V(x)$ as determined by the model of the system (usually it has a polynomial construction), an explicit form of $M(x)$ can be found by assuming that the dynamics of the system are frozen in the ordered/dense state (small M) and that fluctuations are large in the disordered/diluted one (large M). In general, $M(x)$ appears in a coarse-grained derivation of master equation [5,13] and can be a function of the spatial derivatives of the field. However, in this paper we consider a nonconservative situation, assuming M to be only a function of the field x. The noise term ζ is taken to be Gaussian with $\langle\zeta\rangle = 0$ for which the fluctuation dissipation relation holds:

$$\langle\zeta(\mathbf{r}, t)\zeta(\mathbf{r}', t')\rangle = 2\sigma_0^2 M(x)\delta(t - t')\delta(\mathbf{r} - \mathbf{r}'). \tag{3}$$

Due to the satisfaction of equation (3), the noise term in equation (1) represents internal fluctuations with an intensity σ_0^2 reduced to the bath temperature.

Stationary properties of the system can be studied with a help of stationary distribution $\mathcal{P}_{st} = \mathcal{P}([x(\mathbf{r})], t \to \infty)$, obtained as a solution of the corresponding Fokker-Planck equation. The standard way to obtain the Fokker-Planck equation for spatially extended systems requires the use of discrete space calculations to avoid the occurrence of singularities [5,11,14]. Considering the system in a d-dimensional, square lattice with a mesh size ℓ, instead of as the continuous equation (1) we get a set of ordinary differential equations

$$\frac{dx_i}{dt} = -M_i\frac{\partial F}{\partial x_i} + \sqrt{M_i}\tilde{\zeta}_i(t), \tag{4}$$

where $x_i(t) \equiv x(\mathbf{r}_i, t)$, $M_i \equiv M(x_i)$ and $\langle\tilde{\zeta}_i(t)\tilde{\zeta}_j(t')\rangle = 2\sigma^2\delta_{ij}\delta(t - t')$, $\sigma^2 = \sigma_0^2/\ell^d$, and index $i = 1 \ldots N$ labels the cells. Using the discrete approximation of the gradient $|\nabla x|^2$, one can rewrite the free energy as follows

$$F = \sum_{i=1}^{N}\left[V_i + \frac{D}{4d\ell^2}\sum_{j\in nn^+(i)}(x_j - x_i)^2\right], \tag{5}$$

where $nn^+(i)$ indicates nearest neighbors in the positive direction of each axis.

Following the standard approach and assuming the Stratonovich interpretation, the corresponding Fokker-Planck equation takes the form [15]

$$\frac{\partial P(\{x_i\}, t)}{\partial t} = \sum_i \frac{\partial}{\partial x_i}\left(M_i\frac{\partial F}{\partial x_i} - \frac{\sigma^2}{2}\frac{\partial M_i}{\partial x_i} + \sigma^2\frac{\partial}{\partial x_i}M_i\right)P(\{x_i\}, t). \tag{6}$$

Assuming no flux conditions, in the stationary case we arrive at the distribution function

$$P_{st}(\{x\}) \propto \exp\left(-U_{eff}(\{x\})/\sigma^2\right),$$
$$U_{eff}(\{x\}) = F(\{x\}) + \frac{\sigma^2}{2}\sum_{i=1}^{N}\ln M_i; \tag{7}$$

in continuum space it takes the Boltzmann-Gibbs form

$$\mathcal{P}_{st}[x] \propto \exp\left(-\frac{\mathcal{U}_{eff}[x]}{\sigma^2}\right) \tag{8}$$

where the effective internal energy functional is

$$\mathcal{U}_{eff}[x] = \mathcal{F}[x] + \sigma^2 S_{eff}[x]. \quad (9)$$

Here we introduce the effective entropy

$$S_{eff}[x] = \frac{1}{2}\int d\mathbf{r} \ln M(x) \quad (10)$$

as defined through the mobility $M(x)$. Because the free energy is not changed, the system states will be determined solely by entropy variations, as caused by the multiplicative character of the noise.

To investigate how entropy variations control the ordering processes, we use the Weiss mean field analysis technique. In the framework of mean field theory, the gradient term in equation (5) can be rewritten as follows:

$$\nabla^2 x \to \frac{2d}{\ell^2}(\eta - x)^2, \qquad \eta \equiv \langle x \rangle. \quad (11)$$

As a result, the effective potential acquires a dependence on an unknown mean-field value η:

$$U_{eff}(x;\eta) = V(x) + \frac{D}{2\ell^2}(\eta - x)^2 + \frac{\sigma^2}{2}\ln M(x). \quad (12)$$

The value η can be calculated as the solution of the self-consistency equation

$$\eta = \int x P_{st}(x;\eta)dx \equiv \Phi(\eta), \quad (13)$$

where

$$P_{st}(x;\eta) = Z^{-1}(\eta)\exp(-U_{eff}(x;\eta)/\sigma^2). \quad (14)$$

Here, Z satisfies the normalization condition, and η plays the role of the order parameter.

From a physical viewpoint, the solution $\eta = 0$ defines the disordered homogeneous phase, and the corresponding distribution function is symmetrical with respect to the origin $x = 0$. If the distribution function is asymmetrical, then the order parameter takes on a nontrivial value, $\eta \neq 0$, and the system is ordered. To solve equation (13) the standard Newton-Raphson procedure is used. One should note that the right hand side of the function $\Phi(\eta)$, formally, can intersect the left hand side of the function η more than once. A number of intersections gives an equal number of roots of the equation $\eta = \Phi(\eta)$ at the related values of the order parameter η. Generally, the number of roots depends on the form of the function $U_{eff}(x;\eta)$ and the related construction of the normalization constant $Z(\eta)$. In the case under consideration, if the local potential $V(x)$ is of symmetrical form, i.e. $V(-x) = V(x)$, then two equivalent oppositely signed nontrivial solutions $+\eta$ and $-\eta$ appears for a particular range of the control parameters. If the potential $V(x)$ is asymmetrical or shifted with respect to the origin $x = 0$, then one can expect more than one different solutions $\eta \neq 0$.

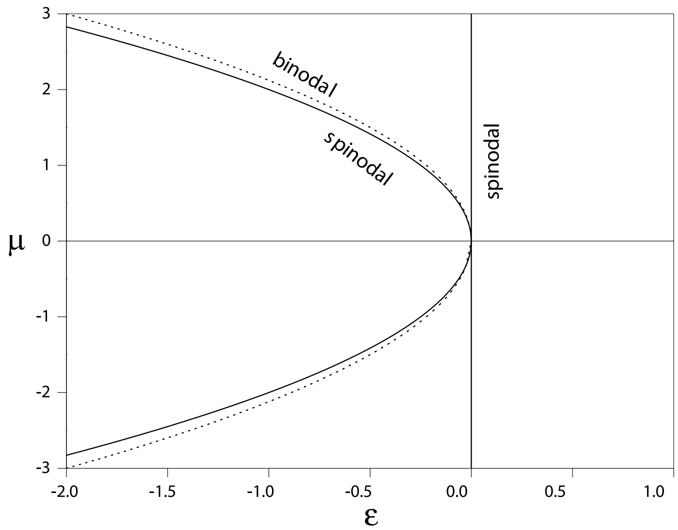

Fig. 1. Phase diagram for the system described by a local potential as in equation (15).

3 Noise induced phase transitions

To proceed, let us assume that in the deterministic regime the system states are characterized by three different stationary values $x_0^{(1)}$, $x_0^{(2)}$ and $x_0^{(3)}$, with $x_0^{(i)} \neq x_0^{(j)}$, $i \neq j$. From a physical viewpoint, the field x can be interpreted as the concentration or as the phase field which evolves according to the deterministic force $f(x) = -\sum_i (x - x_0^{(i)}) = -V'(x)$. In our approach we use the local potential in the form

$$V(x) = \frac{x^4}{4} + \frac{\mu}{3}x^3 - \frac{\varepsilon}{2}x^2, \quad (15)$$

where μ and ε are constants that control the system dynamics. The stationary states are thus: $x_0^{(1)} = 0$, $x_0^{(2)} = -a$, $x_0^{(3)} = b$. In our calculations, we use the more convenient parameters $\mu = a - b$, $\varepsilon = ab$. Following the standard approach of the catastrophe theory, spinodals are defined by the dependencies $\mu_0^s = 0$, $\mu_\pm^s = \pm 2\sqrt{-\varepsilon}$, and binodals are given by $\mu_\pm^b = \pm 3/2\sqrt{-2\varepsilon}$ (see Fig. 1).

We assume the mobility $M(x)$ to be of the form

$$M(x) = \frac{1}{1 + \alpha x^2}, \quad \alpha \geq 0. \quad (16)$$

Variations in the parameter α allow one to consider additive ($\alpha = 0$) or multiplicative ($\alpha \neq 0$) noises, separately.

To show that the above fluctuations do not lead to instability of the disordered state $\langle x \rangle \equiv \eta = 0$, we perform a linear stability analysis. Averaging equation (1) with functions (15) and (16), the linear evolution equation for the first statistical moment reads

$$\frac{\partial \langle x \rangle}{\partial t} = (\varepsilon - \alpha\sigma^2)\langle x \rangle + \frac{D}{2d}\Delta\langle x \rangle. \quad (17)$$

In the case of a monostable local potential ($\varepsilon < 0$) the noise stabilizes the null state, whereas for $\varepsilon > 0$, it leads to the instability of the state $\langle x \rangle = 0$. A more detailed description follows, and is provided using the mean field theory.

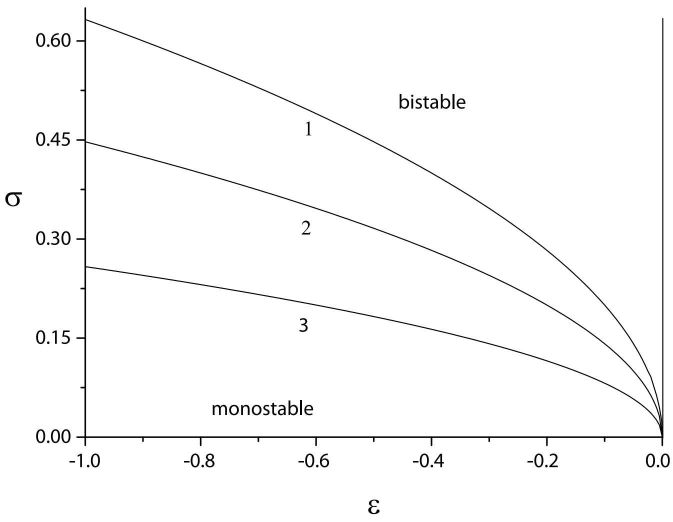

Fig. 2. Phase diagram of noise-induced transitions in a zero-dimensional system ($D = 0$). Curves 1–3 correspond to values $\alpha = 5$, 10, 30.

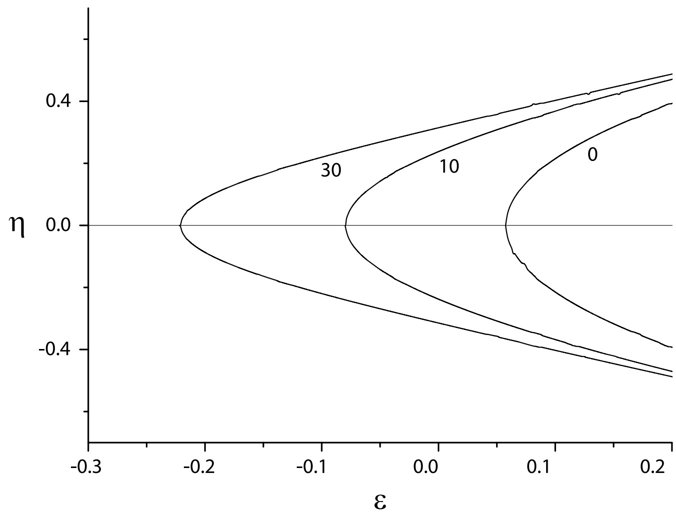

Fig. 3. Order parameter η vs. control parameter ε at $\sigma = 0.2$ and $D = 1.0$. Numbers near curves correspond to values of the parameter α.

3.1 Symmetric potential V(x)

In this subsection, we discuss the example of phase transitions in a symmetric bistable local potential, assuming $\mu = 0$, and using a form $V(x) = x^4/4 - \varepsilon x^2/2$. In the stationary deterministic regime ($\sigma = 0$) the system moves into one of the two possible minima $x^{\pm}_{(0)} = \pm\sqrt{\varepsilon}$.

The influence of noise also merits discussion. A representation of noise-induced transitions ($D = 0$) is presented as a phase diagram in Figure 2. The simplest case of additive noise corresponds to the choice $\alpha = 0$. Under this condition, relaxation processes occur with the same relaxation time $M^{-1} =$ const. in the vicinity of both dilute and dense phases.

As a result, in this case the effective internal energy U_{eff} has the same form as the free energy $\mathcal{F}$. Additive fluctuations smear the stationary distribution near its extrema and do not change the system states as determined by deterministic analysis. In the case of multiplicative fluctuations $\alpha \neq 0$, the noise can play a crucial role. To investigate noise-induced transitions, we analyze two conditions: $dU_{eff}(x)/dx = 0$ and $d^2U_{eff}(x)/dx^2 = 0$. The first of these two cases gives a number of extrema in the effective potential. A concurrent execution applying both conditions allows one to find critical values for the control parameters. The corresponding phase diagram for noise-induced transition is shown in Figure 2. It is seen that in the case of the monostable potential $V(x)$, at $\varepsilon < 0$ an increase in the noise amplitude σ leads to a noise-induced transition when the number of extrema of the effective potential U_{eff} is changed. The critical value σ decreases with an increase in the relaxation parameter α. It is an important result that the bifurcation occurs at $\varepsilon < 0$, corresponding to the monostable form of the local potential $V(x)$. The number of extrema of U_{eff} is changed only by the x-dependent mobility M. In such a case, the above result indicates that noise-induced transitions are driven by the entropy variability.

Next, let us discuss the situation of noise-induced phase transitions assuming $D \neq 0$. An additive noise influence is outside the scope of this study, but we will use it to compare the results obtained for multiplicative noise ($\alpha \neq 0$). The behaviour of a system with multiplicative noise and a monostable, quadratic local potential $V(x)$, was studied in reference [10]. It was shown that an increase in the noise intensity σ^2 leads to an ordering phase transition. Here we address the case where the principal role is played by the biquadratic nonlinearity in the local potential $V(x)$. Moreover, we aim to study the influence of the dissipation parameter α on phase transitions in the system. Towards this aim, let us examine the order parameter dependence on the control parameter ε at different values of the coefficient α. The symmetry of the effective potential is broken due solely to the spatial coupling. Hence, positive and negative solutions of the self-consistency equation differ in sign. As shown in Figure 3, the behavior of the order parameter is predictable at $\varepsilon > 0$ when both the local and effective potentials are bistable. Comparing curves with different values of the parameter α, it is seen that an increase in α leads to ordering dynamics with negative values ε that correspond to the monostable local potential V.

The behaviour of the order parameter η versus noise amplitude σ is shown in Figure 4 for different values of the control parameter ε. Here we plot only the positive solutions of equation (13). It is seen that at positive values of the control parameter ε (curve 1) the system is ordered up to a critical threshold of the noise amplitude σ. If we decrease the control parameter (curve 3), then the system becomes ordered within a fixed range of the noise amplitude σ. This result signifies that an increase in the noise amplitude leads to the formation of the ordered state at small σ, whereas at large σ, the fluctuations destroy the order, thus restoring the symmetry of the stationary distribution. To describe the mechanism of the above reentrant phase transition, let us discuss the behaviour of the

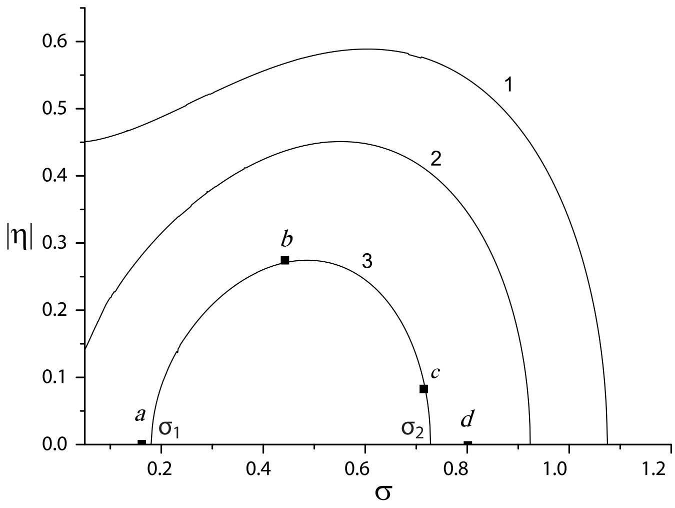

Fig. 4. Order parameter η vs. noise amplitude σ at $D = 1.0$ and $\alpha = 30$. Curves 1, 2, 3 correspond to $\varepsilon = 0.2$, 0.0, –0.2, respectively. At $\varepsilon < 0$ (curve 3) the ordered state is observed inside the range $\sigma \in [\sigma_1, \sigma_2]$.

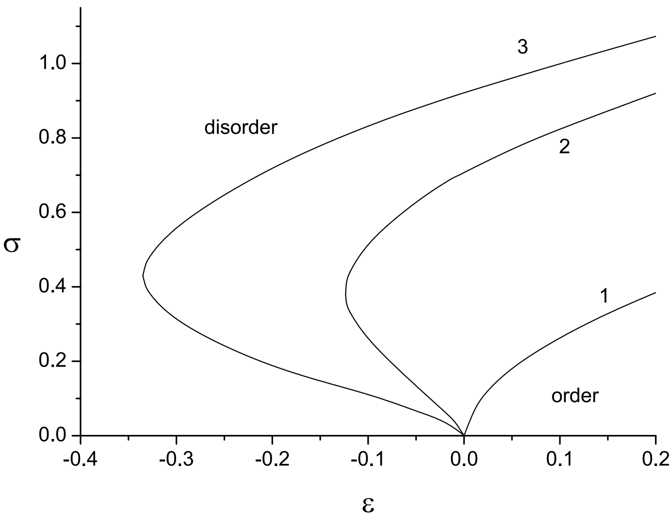

Fig. 6. Phase diagram of system with internal noise for $D = 1.0$ and for different values of the parameter α. Curves 1, 2 and 3 correspond to the cases of additive noise with $\alpha = 0$ and multiplicative noise with $\alpha = 10$ and 30, respectively.

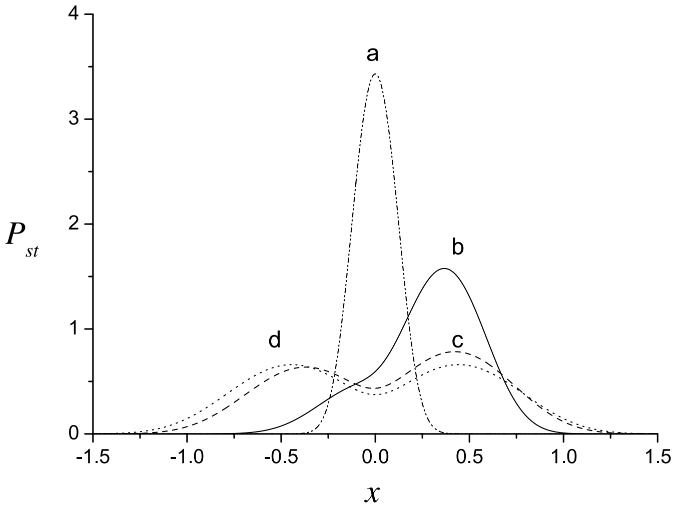

Fig. 5. Stationary probability density $P_{st}(x;\eta)$ for different values of η and noise amplitude σ. Curves a–d correspond to points (η magnitudes) in Figure 4.

stationary distribution at different magnitudes of the order parameter η. Every solution η of the self-consistency equation defines the form of the corresponding probability density function shown in Figure 5. It is seen that in the case of a monostable potential $U_{eff}(x)$ with $\varepsilon < 0$ and small values of the noise amplitude σ, the system is disordered (see point a in Fig. 4). The corresponding probability density function (curve a in Fig. 5) has a symmetrical form with a maximum centered at the origin $x = 0$. If the noise amplitude is increased, then the system becomes ordered (point b in Fig. 4). The corresponding probability density function (curve b in Fig. 5) is characterized by a shift of the single extremum in the positive direction of the x-axis. A further increase in σ (see point c in Fig. 4) leads to a genuine noise-induced transition, as the stationary distribution has a bimodal form (see curve c in Fig. 5). Because the order parameter takes on a nontrivial value, the bimodal distribution is asymmetric. At the point d, the system is disordered; the corresponding distribution is bimodal but has a symmetrical form (see curve d in Fig. 5). Therefore, in the case of a symmetrical local potential, the internal multiplicative noise leads to a reentrant phase transition. During this phase transition, the number of extrema of the probability density function is changed. This implies the following mechanism for such a reentrant phase transition: (i) below the first threshold σ_1 the system is disordered, the distribution is unimodal and symmetrical with respect to the origin; (ii) inside the domain $\sigma \in [\sigma_1, \sigma_2]$ the system is ordered, characterized by an asymmetrical distribution, and the number of extrema of the distribution is changed due to a genuine noise-induced transition; (iii) after the second threshold σ_2 is passed, the distribution is bimodal and symmetrical with respect to the origin.

Next, let us discuss the phase diagram shown in Figure 6. Here we plot the noise amplitude σ versus the control parameter ε at different values of the coefficient α. As Figure 6 shows, an increase in the noise amplitude σ at destroys the ordered state with $\eta \neq 0$ $\alpha = 0$ (lower curve) and $\varepsilon > 0$, and as well, the symmetry of the effective potential U_{eff} is restored ($\eta = 0$). This is a typical example of disorder-creating phase transitions caused by additive noise ($M =$ const.), as are observed in thermodynamic systems. In the case of x-dependent mobility, the behavior of the system is more complicated. In the case of a double-well potential ($\varepsilon > 0$), the multiplicative noise leads to disordering phase transitions, as expected. One needs to point out that in the presence of thermal fluctuations alone, the system can be only become ordered for negative values of the control parameter ε. At small noise amplitudes and $\varepsilon < 0$, the system is in the disordered

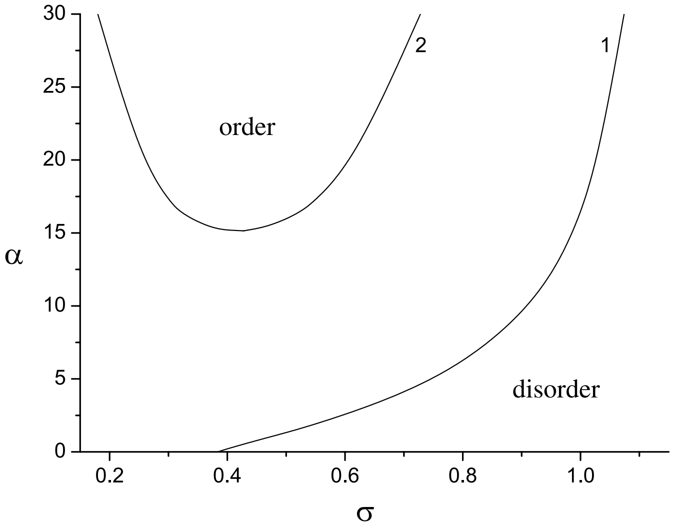

Fig. 7. Phase diagram of the system for $D = 1.0$ and different values of the control parameter ε. Curves 1 and 2 correspond to $\varepsilon = 0.2$ and –0.2, respectively.

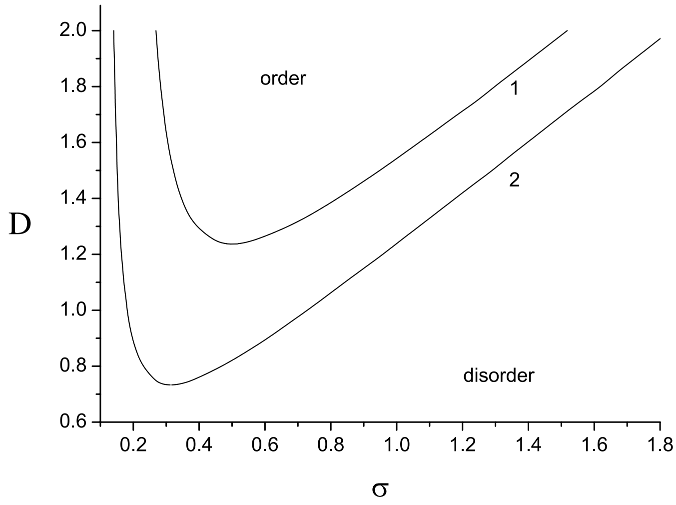

Fig. 8. Phase diagram of the system for $\varepsilon = -0.2$ (curves 1 and 2 correspond to values $\alpha = 10$ and 30, respectively).

state, $\eta = 0$. An increase in σ leads to ordering dynamics at $\sigma_1 \leq \sigma \leq \sigma_2$. A further increase in σ destroys the ordered state, restoring the symmetry of the system. Hence, the multiplicative thermal noise yields reentrant phase transitions if the system is characterized by a monostable potential $V(x)$. It is principally important that the disordering processes — and hence the reentrance — are due to the biquadratic nonlinearity in $V(x)$, this despite the fact that the ordering situation is related to the quadratic term in V, as sown in study [10].

Next, we present a phase diagram to demonstrate the manner in which thermal fluctuations can lead to an ordering of the system (see Fig. 7) It is seen that in the case of a bistable local potential $\varepsilon > 0$, an increase in σ leads to a disordering processes. Additionally, an increase in the temperature destroys the ordered phase, thus restoring the symmetry of the probability density function. In the case of a monostable local potential ($\varepsilon < 0$) with an x-dependent mobility, one observes the reentrant ordering phase transition with an increase in the temperature σ^2.

In Figure 8 we present a phase diagram in the (D, σ) plane to show the influence of the parameter α on the bifurcation magnitudes of the spatial coupling intensity D and the noise amplitude σ. One can see that in the case of multiplicative noise ($\alpha \neq 0$) a doubly bounded domain of ordered phase appears where a reentrant transition is observed. An increase in the relaxation parameter α decreases the critical values of the spatial coupling intensity D and extends the range of the noise amplitude in which the ordered phase exists.

Therefore, in a symmetrical system, the thermal multiplicative noise leads to: (i) noise-induced transitions with a changing number of extrema in the probability density function; (ii) a shift of the critical point for noise induced phase transitions; (iii) a reentrance phenomenon that occurs for a monostable local potential V due to nonlinearities of higher (even) order in $V(x)$.

3.2 Asymmetric potential V(x)

In this subsection, we present a typical scenario of noise-induced phase transitions in the systems characterized by an asymmetric local potential (15). Considering the change in the topology of the bifurcation curves as shown in Figure 1, one can say that the presence of a cubic term ($\mu \neq 0$) in the local potential V leads to the well-known phenomenon of a phase transition with a symmetry-breaking term (see Fig. 9a, cf. with Fig. 3). In this situation, an increase in $|\mu|$ shifts the bifurcation point for solutions $\eta < 0$ toward large, and positive ε values. At large $|\mu|$, a hysteresis-like behavior in the semiaxis $\eta > 0$ is observed.

In continuing the investigation, we present a self-organization situation, considering only positive solutions η which have a physical meaning. To this end, we assume that the system can be in one of two minima placed at $x_0^{(0)} = 0$ or x corresponds to a maximum in $V(x)$. Choosing $|a|, b \leq 1$ in equation (15), we localize the above extrema to the semiaxis $x \geq 0$. Continuing the study of noise-induced phase transitions, we investigate the behaviour of the order parameter for $\varepsilon < 0$ and $0 < |\mu| \leq 2$. In Figure 9b, we present solutions of the self-consistency equation at $\mu = -2$ and for different values of the noise amplitude σ and the parameter α. It is seen that an increase in the noise amplitude σ or the relaxation parameter α suppresses the formation of a hysteresis loop and shifts the position of the bifurcation points.

We now discuss the order parameter behaviour as a function of μ for different values of the noise amplitude. The behaviour of the phase transitions is intuitive in the case of additive noise ($\alpha = 0$), see Figure 10a. Here, at small noise amplitudes, the system undergoes phase transitions in the following manner. An increase in the control parameter μ (from a negative value towards zero) suppresses the order in the system until $\mu = 0$, at which point the system is characterized by the trivial value of the order parameter $\eta = 0$. With a further increase in μ,

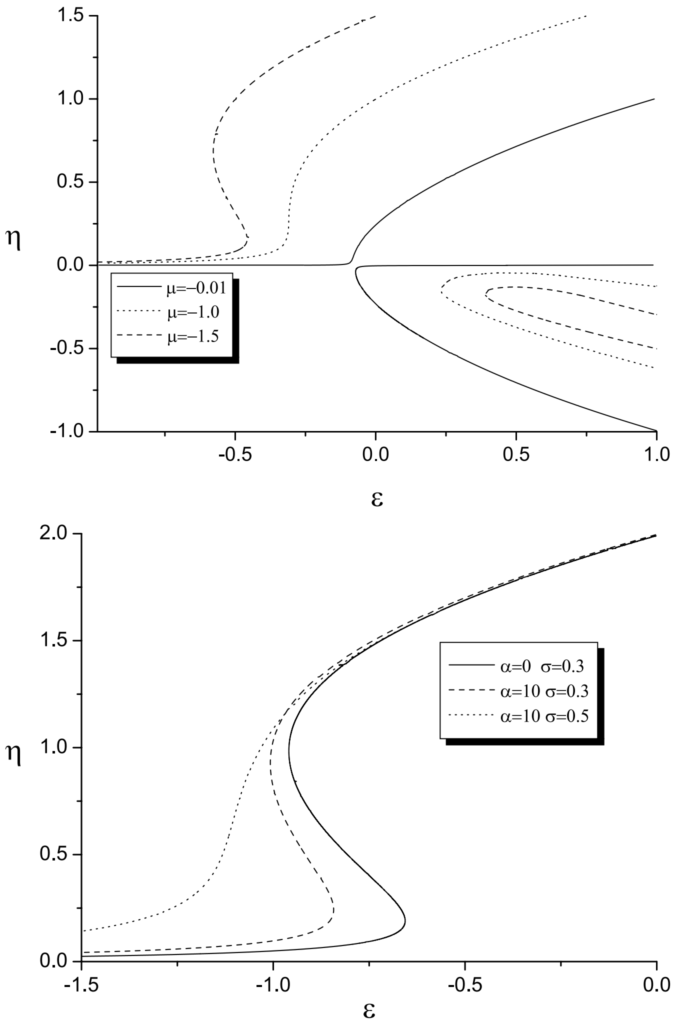

Fig. 9. Order parameter η vs. ε for $D = 1.0$ (a) $\alpha = 10$, curves are plotted for different values of the control parameter μ; (b) $\mu = -2.0$, curves are plotted for different values of the noise amplitude σ and the parameter α.

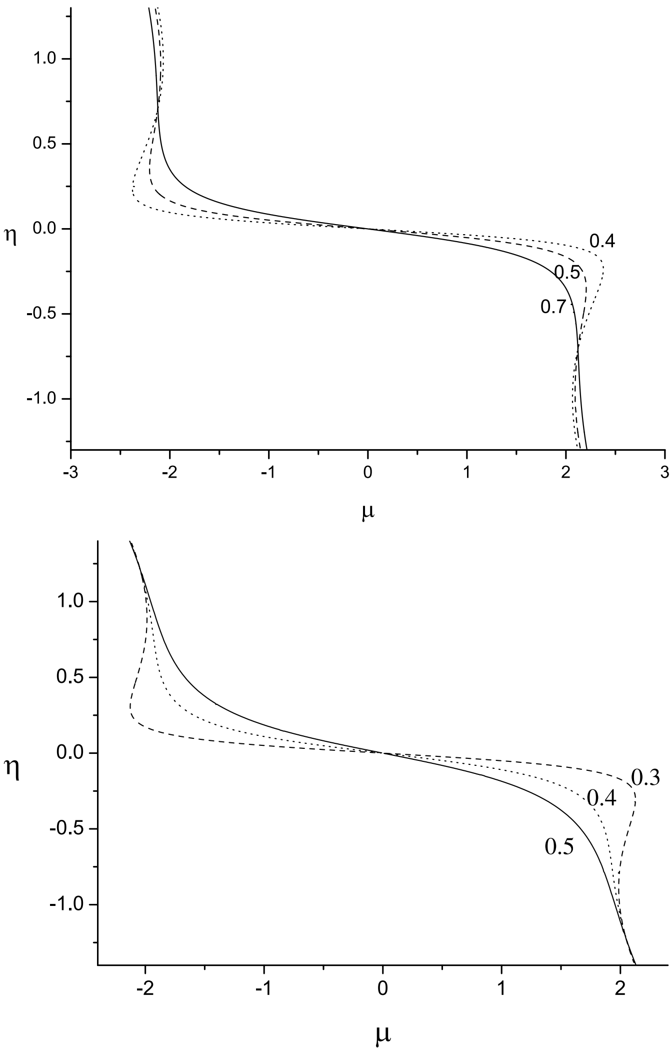

Fig. 10. Order parameter η vs. control parameter μ for $D = 1$ and $\varepsilon = -1$. Numbers near curves correspond to values of the noise amplitude σ: (a) order parameter behaviour in the case of additive noise influence ($\alpha = 0$); (b) order parameter behaviour in the case of multiplicative noise at $\alpha = 30$.

the system moves into an ordered state with a negative value adopted by the order parameter. The corresponding curves in Figure 10 all have point symmetry. An increase in the noise amplitude σ completely suppresses bifurcations, and as a result, we obtain the classical situation of continuous variation in the order parameter. At $\alpha \neq 0$ the sensitivity of the system to the influence of noise increases. As Figure 10b shows, despite the topology of the bifurcation curves being unchanged, one can see that bifurcation processes disappear at small values of the noise intensity. As in the previous case, an increase in the noise amplitude suppresses the step-like behaviour of the order parameter when μ varies. The above results illustrate the shift of the bifurcation points due only to the control parameter, but do not demonstrate the crucial role of multiplicative noise.

To determine the influence of multiplicative noise on the ordering processes, we next discuss the behaviour of the order parameter η in response to the noise amplitude σ. In our treatment, we take $\mu = -2$ and $\varepsilon = -1$ (spinodal in Fig. 1) and vary the noise amplitude σ, the relaxation parameter α and the spatial coupling intensity D. From this simplified treatment, one can conclude that variation in D will only change the positions of the bifurcation points. One can see that the spatial coupling term only appears in the free energy, and that the coefficient D does not enter into any other terms in U_{eff}. However, variations in the noise intensity σ^2 and α can lead to more complicated effect on the effective potential U_{eff} and hence one can expect bifurcations when σ or α varies. Let us consider solutions of the self-consistency equation as shown in Figure 11. As previously mentioned, at $\alpha \neq 0$ and $D \neq 0$ the system is asymmetric. As the local potential is initially asymmetric, the order parameter has a nontrivial value at $\alpha \neq 0$ and at $D = 0$. At $\alpha = 0$ and $D \neq 0$, we obtain the additional term which breaks the symmetry of U_{eff}. From Figure 11a, one can see that at $\alpha \neq 0$ and $D = 0$ (curve 1) the order parameter is always positive regardless of the choice of μ and ε. At $D \neq 0$, spatial coupling

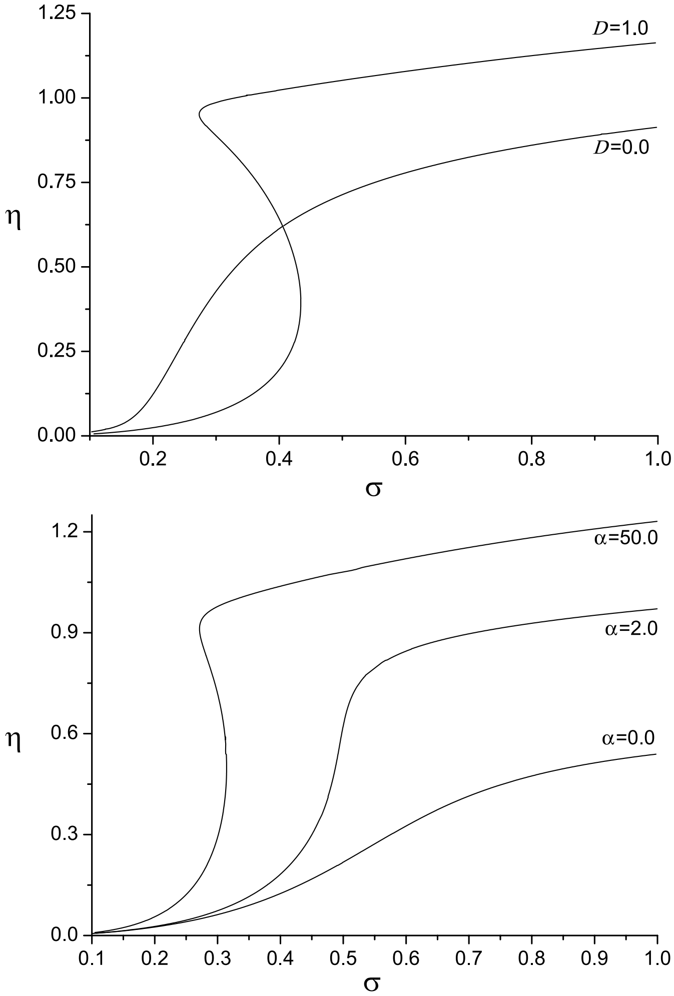

Fig. 11. Order parameter η vs. noise amplitude σ for $\mu = -2$ and $\varepsilon = -1$: (a) $\alpha = 5.0$, curves are plotted for different values of interaction intensity D; (b) $D = 0.7$, curves for different values of the parameter α.

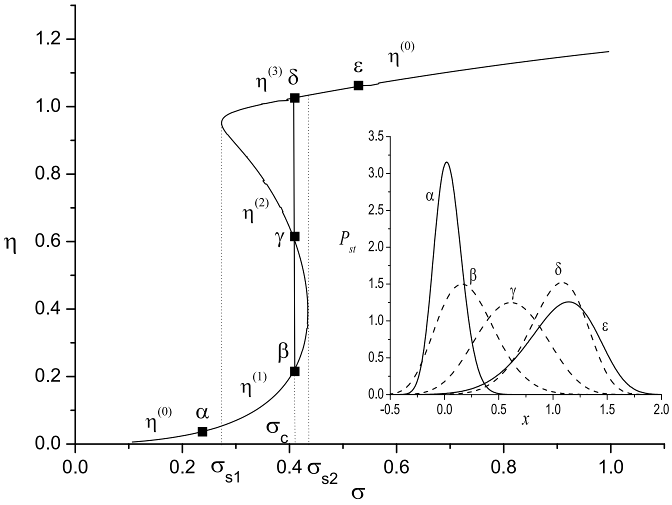

Fig. 12. Order parameter dependence of the positions of critical values of the noise amplitude. These define the positions of the spinodals σ_{s1}, σ_{s2} and the binodal σ_c in the phase diagram of Figure 13. Insert: Typical form of distribution functions for different values of the order parameter at the points indicated on the $\eta(\sigma)$ curve.

promotes the S-like behavior of the order parameter. This result indicates that except for the stable solutions of the self-consistency equation — shown as the outer branches in Figure 11 — we obtain an unstable solution, as shown by the middle curve. An increase in the relaxation parameter α yields the same hysteresis-like effect for the order parameter (see Fig. 11b) as when the temperature varies.

To study the phase diagrams corresponding to this set of conditions, we calculate the dependence of the spinodals that correspond to the positions of the points σ_{s1}, σ_{s2} in $\eta(\sigma)$ (see Fig. 12) when the condition $d\eta(\sigma)/d\sigma = \infty$ is satisfied. To find the binodals, we have used the following algorithm. In the case of a unique solution $\eta^{(0)}$ of the self-consistency equation (at $\sigma \notin [\sigma_{s1}, \sigma_{s2}]$), then the system is characterized by a unique probability density function $P_{st} = P_{st}(x; \eta^{(0)})$ which has a single maximum centered on $\eta^{(0)}$. In the case of three possible solutions of equation (13) at $\sigma \notin [\sigma_{s1}, \sigma_{s2}]$ then we have three corresponding probability density functions. Each probability density is scaled to the corresponding magnitude of the order parameter and is centered in its vicinity, i.e.: $P_{st}^{(i)} = P_{st}(x; \eta^{(i)})$, where $i = 1, 2, 3$ denotes different solutions of equation (13), as shown in Figure 12. Each distribution has a unique maximum with different magnitudes and characterized by a unique dispersion $\langle (x - \langle x \rangle)^2 \rangle$. Using the ergodicity theorem, one can conclude that the binodal should correspond to the values σ_c in the (σ, α) plane (see Fig. 13) if the corresponding maximal values of the above distributions and the related dispersions are identical. At last, as follows from the phase diagrams shown in Figure 13, a hysteresis loop can be formed if the parameter α is varied. Finally, we present the phase diagram in the plane (σ, D) (see Fig. 13b). Comparing the corresponding phase diagram of Figure 8, one can say that just as in the case of a symmetrical local potential, with an increase in the parameter α, the critical values of both the spatial interaction intensity D and the noise amplitude σ decrease. In other words, the ordered phase at large α can be formed at small values of the spatial interaction intensity D. Following catastrophe theory, one can conclude that due to the asymmetric form of the local potential, the above phase transitions are of the first kind.

4 Conclusions

We have considered two possible generalizations of entropy-driven phase transitions in physical systems with a relaxation flow and a field-dependent kinetic coefficient. It is shown that the internal multiplicative noise induces the reentrant behaviour of the order parameter in the case of a monostable, symmetrical local potential. If the local potential has an asymmetric from, then hysteresis in the phase transitions are observed.

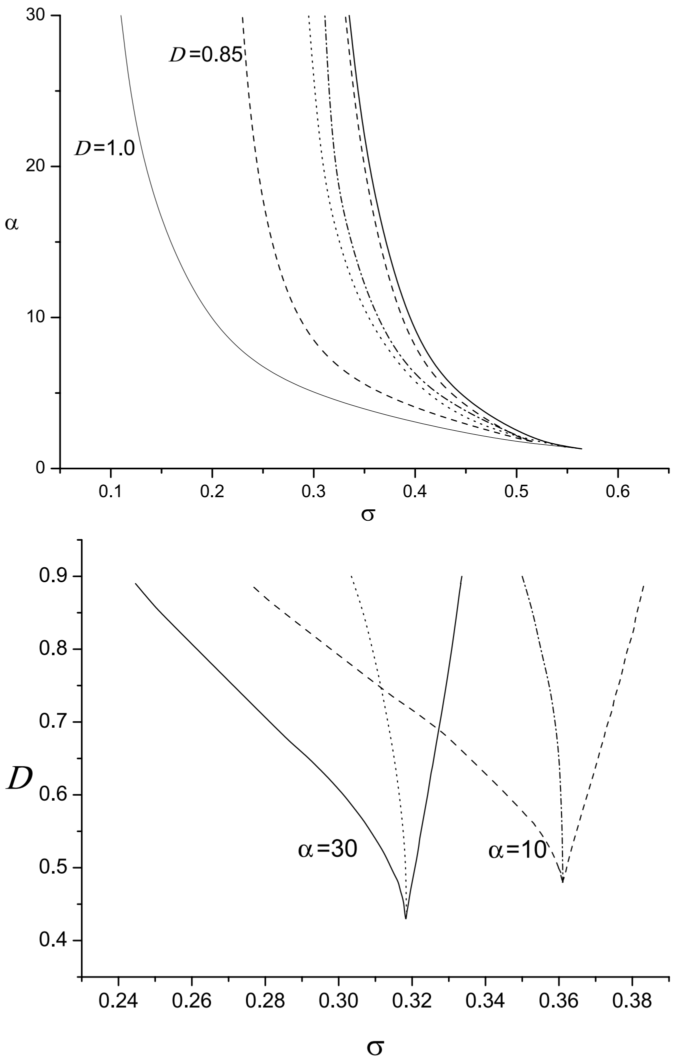

Fig. 13. Phase diagrams for system with a nonsymmetrical local potential and for $\mu = -2$, $\varepsilon = -1$: (a) solid and dotted lines are spinodals and binodal respectively, for $D = 1.0$; dashed and dash-dotted lines are spinodals and binodal respectively, for $D = 0.85$; (b) solid and dotted lines are spinodals and binodal for $\alpha = 30$, dashed and dash-dotted lines are spinodals and binodal for $\alpha = 10$.

The mechanism of the reentrant phase transition is as follows. At small noise intensities, the system is in the disordered state, which is characterized by a unimodal distribution symmetrical with respect to the origin. With an increase in the noise amplitude, the symmetry of the stationary distribution is broken due to the nonlinearity of the system and of the spatial coupling. As a result, the order parameter takes on a nontrivial value. A further increase in the noise amplitude leads to a genuine noise-induced transition, wherein an additional maximum in the stationary distribution appears. Despite the fact that the distribution is still asymmetric, the system is nevertheless ordered. At large noise intensities, the symmetry of the bimodal distribution is restored, which trivializes the value of the order parameter. The above processes can be controlled by variations in the parameter that governs the relaxation flow. In the case of an asymmetric local potential, a system with a field-dependent kinetic coefficient demonstrates a hysteresis-like behaviour. The positions of the corresponding spinodals and binodals depend on the noise intensity and on the relaxation parameter.

The case of a symmetric potential can be considered as a model for reentrant behavior in magnetic systems such as Roshelt salt, where temperature variations leads to reentrant behaviour in the magnetization and in the lattice structure. Models of systems with an asymmetric potential, as considered in the work, are used to describe ordering processes in phase field theory. We have shown that a relaxation flow — with a filed-dependent kinetic coefficient which leads to entropy variations — serves as an additional mechanism for ordering processes in thermodynamic systems. In this Letter, only the main results of one type of noise influence are presented. More detailed analysis should be done to investigate the role of correlations between both internal and external fluctuating sources. Another prospective direction is the study of the effects of phase separation in the above systems but with conservative dynamics.

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